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THERMAL STRESSES IN A HEAT-SENSITIVE SPHERE

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A solution is obtained for the quasistatic problem of thermoelasticity for a heat-sensitive sphere heated by a heat flux. Thermal stresses are investigated in a steel sphere.

Let us examine an isotropic elastic sphere of radius R, free from external load but subjected to sudden heating by a heat flux of constant density q. The initial temperature of the sphere is zero. All the physicomechanical characteristics of the material except the coefficient ν are functions of the temperature.

For many materials [1] the temperature dependences of the heat conduction $\lambda_t(t)$ and volume specific heat $c_V(t)$ coefficients are identical in nature, whereupon their coefficient of thermal diffusivity is $a = \lambda t(t)/c_V(t) = \mathrm{const}$. Then by using the Kirchhoff variable

$$\vartheta^*(t^*) = \int_0^{t^*} \lambda_t^*(\xi)\, a\xi \tag{1}$$

the nonlinear heat-conduction problem is linearized. We consequently arrive at a boundary-value problem for the Kirchhoff variable:

$$\rho^{-2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\vartheta^*}{\partial\rho}\right) = \frac{\partial\vartheta^*}{\partial\,\mathrm{Fo}}, \tag{2}$$

$$\left.\frac{\partial\vartheta^*}{\partial\rho}\right|_{\rho=0} = 0, \quad \left.\frac{\partial\vartheta^*}{\partial\rho}\right|_{\rho=1} = \mathrm{Ki}\, S_+(\mathrm{Fo}), \tag{3}$$

$$\vartheta^*(\rho,\ 0) = 0, \tag{4}$$

whose solution has the form [2]

$$\vartheta^* = \mathrm{Ki}\left[\ 3\mathrm{Fo} - \frac{3-5\rho^2}{10} - \sum_{n=1}^{\infty}\frac{2\sin\mu_n\rho}{\mu_n^3\rho\cos\mu_n}\exp(-\mu_n^2\ \mathrm{Fo})\ \right], \tag{5}$$

where $t^* = t/t_0$; $\rho = r/R$; $\mathrm{Fo} = a\tau/R^2$; $\lambda_t^*(t^*) = \lambda_t(t)/\lambda_t^{(0)}$; $\mathrm{Ki} = qR/\lambda_t^{(0)}t_0$ is the Kirpichev criterion; μ_n are the roots of the characteristic equation $\tan\mu = \mu$.

Knowing the expression for the Kirchhoff variable and the temperature dependence of the heat-conduction coefficient, we can determine the temperature field in the sphere from the relationship (1).

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To determine the thermal stresses occurring in the sphere, we use the formulas [1]

$$\sigma_r = G^*(t^*)\left[(1-\nu)\frac{\partial u}{\partial \rho} + 2\nu\frac{u}{\rho} - (1-\nu)\Phi^*\right],$$
$$\sigma_\varphi = \sigma_\Theta = G^*(t^*)\left[\nu\frac{\partial u}{\partial \rho} + \frac{u}{\rho} - (1-\nu)\Phi^*\right], \tag{6}$$

where the displacement u satisfies the equation

$$\frac{\partial}{\partial \rho}\left[\rho^{-2}\frac{\partial}{\partial \rho}(\rho^2 u)\right] + \frac{1}{G^*(t^*)}\frac{\partial G^*}{\partial \rho}\left[\frac{\partial u}{\partial \rho} + \frac{2\nu}{1-\nu}\frac{u}{\rho} - \Phi^*\right] = \frac{\partial \Phi^*}{\partial \rho}. \tag{7}$$

Here

$$u = \frac{U}{R\alpha_t^{(0)}t_0},\quad \sigma_i = \frac{1-2\nu}{2G_0\alpha_t^{(0)}t_0}\sigma_{ii}\ (i = r,\ \varphi,\ \Theta).$$
$$\Phi^* = \frac{1+\nu}{1-\nu}\int_0^{t^*}\alpha_t^*(\eta)\,d\eta,\quad \alpha_t^*(t^*) = \frac{\alpha_t(t)}{\alpha_t^{(0)}},\quad G^*(t^*) = \frac{G(t)}{G_0}.$$

We take the boundary conditions in the form

$$u|_{\rho=0} = 0,\quad \sigma_r|_{\rho=1} = 0. \tag{8}$$

To solve the boundary-value problem (6)-(8) we proceed as follows. We approximate the functional dependence $G^*(t^*)$ in its domain of definition to a given accuracy by the expression

$$G^*(t^*) = G_1^* + \sum_{i=1}^{n}(G_{i+1}^* - G_i^*)S_+(t^* - t_i^*), \tag{9}$$

where

$$t_H^* < t_1^* < \ldots < t_n^* < t_K^*,\quad S_+(\xi) = \begin{cases}1, & \xi > 0,\\ 0, & \xi \leqslant 0,\end{cases}$$

is the asymmetric unit function, and (t_H^*, t_K^*) is the temperature interval of the change in shear modulus.

Substituting (9) into (7) and taking into account here that time plays the part of a parameter in (7) while $t^*(\rho, \tau)$ is defined for each specific case, we obtain at an arbitrary fixed time

$$\frac{\partial}{\partial \rho}\left[\rho^{-2}\frac{\partial}{\partial \rho}(\rho^2 u)\right] + \sum_{j=1}^{m_\tau}\frac{G_{q_\tau+j+1}^* - G_{q_\tau+j}^*}{G_{q_\tau+j+1}^*}\left(\frac{\partial u}{\partial \rho} + \frac{2\nu}{1-\nu}\frac{u}{\rho} - \right.$$
$$\left. - \Phi^*\right)\delta_+(t^* - t_{q_\tau+j}^*)\frac{\partial t^*}{\partial \rho} = \frac{\partial \Phi^*}{\partial \rho}, \tag{10}$$

$$t_1^* < \ldots < t_{q_\tau}^* \leqslant \min t^* < t_{q_\tau+1}^* < \ldots < t_{q_\tau+m_\tau}^* <$$
$$< \max t^* < t_{q_\tau+m_\tau+1}^* < \ldots < t_n^*, \tag{11}$$

where

$$\min t^* = \min_{\rho\in[0,1]} t^*(\rho, \tau),\quad \max t^* = \max_{\rho\in[0,1]} t^*(\rho, \tau).$$

Since $t_{q_\tau+j}^*$ $(1 \leqslant j \leqslant m_\tau)$ satisfies (11) while $t^*(\rho, \tau)$ is monotonic for $\rho \in [0, 1]$ and $\tau \in [0, \infty)$ in our case, then the equation

$$t^*(\rho, \tau) - t_{q_\tau+j}^* = 0 \tag{12}$$

has just one prime root $\rho_{q_\tau+j}$ at each fixed time.

If the time is considered as a parameter, the following relationships are valid:

$$S_+[t^*(\rho, \tau) - t_{q_\tau+j}^*] = S_+\left[\operatorname{sign}_+\left(\left.\frac{\partial t^*}{\partial \rho}\right|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right],$$

$$\delta_+[t^*(\rho,\ \tau) - t^*_{q_\tau+j}] = \left|\frac{\partial t^*}{\partial \rho}\right|^{-1}_{\rho_{q_\tau+j}} \delta_+\left[\operatorname{sign}_+\left(\frac{\partial t^*}{\partial \rho}\Big|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right]. \tag{13}$$

In that case, taking account of (13) we have from (10)

$$\frac{\partial}{\partial\rho}\left[\rho^{-2}\frac{\partial}{\partial\rho}(\rho^2 u)\right] + \sum_{j=1}^{m_\tau}\frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{G^*_{q_\tau+j+1}}\left[\frac{\partial u}{\partial\rho} + \frac{2\nu}{1-\nu}\frac{u}{\rho} - \Phi^*\right]\Big|_{\rho_{q_\tau+j}}\operatorname{sign}_+\left(\frac{\partial t^*}{\partial\rho}\Big|_{\rho_{q_\tau+j}}\right)\times$$
$$\times\ \delta_+\left[\operatorname{sign}_+\left(\frac{\partial t^*}{\partial\rho}\Big|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right] = \frac{\partial\Phi^*}{\partial\rho}. \tag{14}$$

Integrating (14), we obtain

$$u(\rho,\ \tau) = \frac{C\rho}{3} + \rho^{-2}[H(\rho,\ \tau) + D] - \sum_{j=1}^{m_\tau}\frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{3G^*_{q_\tau+j+j}}\left(\rho - \frac{\rho^3_{q_\tau+j}}{\rho^2}\right)\sigma|_{\rho_{q_\tau+j}} S_+\left[\operatorname{sign}_+\left(\frac{\partial t^*}{\partial\rho}\Big|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right]. \tag{15}$$

Substituting (15) into (6), we find expressions for the thermal stresses

$$\sigma_r = \frac{2}{3}(1-2\nu)G^*(t^*)\Bigg\{\frac{(1+\nu)C}{2(1-2\nu)} + 3\rho^{-3}[H(\rho,\ \tau) + D] -$$
$$-\sum_{j=1}^{m_\tau}\frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{2G^*_{q_\tau+j+1}}\left(\frac{1+\nu}{1-2\nu} + \frac{2\rho^3_{q_\tau+j}}{\rho^3}\right)\times$$
$$\times\sigma|_{\rho_{q_\tau+j}} S_+\left[\operatorname{sign}_+\left(\frac{\partial t^*}{\partial\rho}\Big|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right]\Bigg\}, \tag{16}$$

$$\sigma_\varphi = \sigma_\theta = (1-2\nu)G^*(t^*)\Bigg\{\frac{(1+\nu)C}{3(1-2\nu)} + \rho^{-3}[H(\rho,\ \tau) + D] - \Phi^*(\rho,\ \tau) -$$
$$-\sum_{j=1}^{m_\tau}\frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{3G^*_{q_\tau+j+1}}\left(\frac{1+\nu}{1-2\nu} - \frac{\rho^3_{q_\tau+j}}{\rho^3}\right)\times$$
$$\times\sigma|_{\rho_{q_\tau+j}} S_+\left[\operatorname{sign}_+\left(\frac{\partial t^*}{\partial\rho}\Big|_{\rho_{q_\tau+j}}\right)(\rho - \rho_{q_\tau+j})\right]\Bigg\},$$

where

$$H(\rho,\ \tau) = \int_0^\rho \xi^2\Phi^*(\xi,\ \tau)\,a\xi;\ \ \sigma|_{\rho_{q_\tau+j}} = \left(\frac{\partial u}{\partial\rho} + \frac{2\nu}{1-\nu}\frac{u}{\rho} - \Phi^*\right)\Big|_{\rho_{q_\tau+j}} =$$
$$= CK_1^{(q_\tau+j)} + DK_2^{(q_\tau+j)} + K_3^{(q_\tau+j)};\ \ K_1^{(q_\tau+j)} = \frac{1-2\nu}{3(1-\nu)}\times$$
$$\times\left[\frac{1+\nu}{1-2\nu} - \sum_{i=1}^{j-1} L_{ij}K_1^{(q_\tau+i)}\right];$$
$$K_2^{(q_\tau+j)} = \frac{2(2\nu-1)}{3(1-\nu)}\left[3\rho^{-3}_{q_\tau+j} + \frac{1}{2}\sum_{i=1}^{j-1} L_{ij}K_2^{(q_\tau+i)}\right];$$
$$K_3^{(q_\tau+j)} = \frac{2(2\nu-1)}{3(1-\nu)}\left[\frac{3}{\rho^3_{q_\tau+j}}H(\rho_{q_\tau+j},\ \tau) + \frac{1}{2}\sum_{i=1}^{j-1} L_{ij}K_3^{(q_\tau+i)}\right];$$
$$L_{ij} = \frac{G^*_{q_\tau+i+1} - G^*_{q_\tau+i}}{G^*_{q_\tau+i+1}}\left(\frac{1+\nu}{1-2\nu} + \frac{2\rho^3_{q_\tau+i}}{\rho^3_{q_\tau+j}}\right),$$
$$(1 \leqslant j \leqslant m_\tau);\ \ \operatorname{sign}_+\eta = 2S_+(\eta) - 1.$$

We find the integration constants C and D from the boundary conditions (8) in the form:

$$C = \frac{a_{23}a_{12} - a_{13}a_{22}}{a_{11}a_{22} - a_{21}a_{12}},\ \ D = \frac{a_{21}a_{13} - a_{23}a_{11}}{a_{11}a_{22} - a_{21}a_{12}}, \tag{17}$$

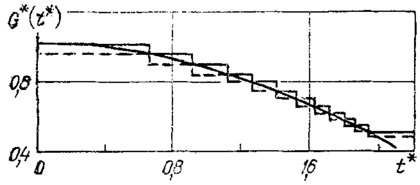

Fig. 1. Temperature dependence of the shear modulus G*(t*) and its approximation by unit functions for different methods of selecting the coefficients G_i^*.

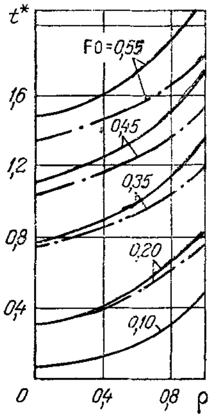

Fig. 2

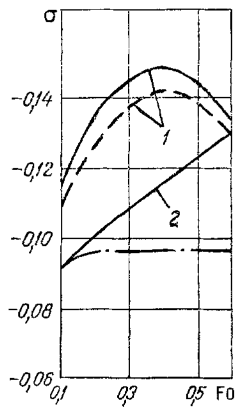

Fig. 3

Fig. 2. Change in the temperature t* as a function of the radius ρ for different values of the Fourier parameter Fo.

Fig. 3. Dependence of the stress $\sigma_\varphi = \sigma_\theta$ on the Fourier parameter Fo on the surface of a sphere (ρ = 1).

where

$$a_{1m} = b_{1m} + Z_{1m};\ b_{11} = 0;\ b_{12} = 1;\ b_{13} = 0;$$

$$a_{2m} = b_{2m} - Z_{2m};\ b_{21} = \frac{1+\nu}{2(1-2\nu)};\ b_{22} = -3;\ b_{23} = -3H(1, \tau);$$

$$Z_{1m} = \sum_{j=1}^{m_\tau} \frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{3G^*_{q_\tau+j+1}} K_m^{(q_\tau+j)} \rho^3_{q_\tau+j} S_+ \left[-\mathrm{sign}_+ \left(\frac{\partial t^*}{\partial \rho}\Big|_{\rho_{q_\tau+j}}\right)\right];$$

$$Z_{2m} = \sum_{j=1}^{m_\tau} \frac{G^*_{q_\tau+j+1} - G^*_{q_\tau+j}}{2G^*_{q_\tau+j+1}} K_m^{(q_\tau+j)} \left(\frac{1+\nu}{1-2\nu} + 2\rho^3_{q_\tau+j}\right) S_+ \left[\mathrm{sign}_+ \left(\frac{\partial t^*}{\partial \rho}\Big|_{\rho_{q_\tau+j}}\right)\right].$$

The temperature fields and the thermal stresses they impose in the steel sphere are investigated by using (5) and (16), where the heat-conduction coefficient and the temperature coefficient of linear expansion of the steel are linear functions while the shear modulus is a quadratic function of the temperature [1, 3]:

$$\lambda_t^* = 1 - k_* t^*,\ \alpha_t^* = 1 + \gamma t^*,\ G^* = 1 - \beta t^{*2}, \tag{18}$$

where t_0 = 333°C; k_* = 0.132; γ = 0.554; β = 0.125. Taking account of (18), the temperature field of the sphere was here determined according to (1) from the formula

$$t^* = \frac{1}{k_*}\left(1 - \sqrt{1 - 2k_*\vartheta^*}\right), \tag{19}$$

while the temperature dependence of the shear modulus (18) was approximated in the temperature range (t_H^*, t_K^*) by the dependence (9), in which

$$n = \left[\left| \frac{\ln G^*(t_H^*) - \ln G^*(t_K^*)}{\ln(1+\varepsilon)} \right| \right],$$

t_i^* are roots of the equation $G^*(t_i^*) - G^*(t_K^*)(1 + \varepsilon)^{n+1-i} = 0$, the coefficients G_i^* are determined by the relationships

$$G_1^* = G^*(t_H^*),\ G_{i+1}^* = G^*(t_K^*)(1+\varepsilon)^{n+1-i},\ i = \overline{1, n}, \tag{20}$$

or

$$G_i^* = G^*(t_K^*)(1+\varepsilon)^{n+1-i},\ i = \overline{1, n+1}, \tag{21}$$

which assured approximation of the real dependence of the shear modulus (18) on the temperature with an excess or deficiency, respectively, ε is the maximal relative deviation of the approximating dependence (9) from the actual (18). Computations were performed for Ki = 1, $\varepsilon = 6\%$, $\nu = 0.4$.

The dependence of the dimensionless shear modulus (18) (solid curve) on the temperature, as well as its approximation to 6% accuracy by (9), in which G_i^* was determined by (20) or (21) (solid and dashed step curves, respectively), are shown in Fig. 1. The change in the temperature t^* (solid curve) as a function of ρ is presented in Fig. 2 for different values of the parameter Fo. The dash-dot lines denote the appropriate values of the temperature in the case of a constant heat-conduction coefficient equal to the reference value.

The dependence of values of the dimensionless circumferential stresses $\sigma_\varphi = \sigma_\Theta$ on the sphere surface is represented in Fig. 3 as a function of the dimensionless parameter Fo. The curves 1 correspond to stresses in the heat-sensitive sphere, when the coefficients G_i^* in (9) were determined by (20) or (21) (solid and dashed curves, respectively). Presented for comparison in Fig. 3 are analogous results when a) only the temperature dependence of the heat-conduction coefficient was taken into account while all the rest were taken constant and equal to appropriate reference values (curve 2) and b) all the characteristics were taken constant and equal to appropriate reference values (dash-dot curve).

Analysis of the results obtained indicates that taking account of the temperature dependence of the material characteristics results both in a quantitative and qualitative change in the thermal stresses; neglecting the temperature dependence of the characteristics results in significant error in the determination of the thermal stresses, which reach $\approx 62\%$ in the case under consideration; utilization of the approximation of the shear modulus by (9), in which G_i^* is determined by (20) or (21), results in a relative error in finding the magnitude of the temperature stresses which does not exceed the accuracy of the approximation.

NOTATION

r, φ, θ, spherical coordinates; $\sigma_{ii} (i = r, \varphi, \Theta)$, stress tensor components in the spherical coordinate system; t, temperature field of the body; τ, time; U, radial displacement; $\lambda_t(t)$, heat-conduction coefficient; $c_v(t)$, volume specific heat; $\alpha_t(t)$, temperature coefficient of linear expansion; G(t), shear modulus; ν, Poisson ratio; q, heat-flux density; t_0, λ_t^0, $c_v^{(0)}$, $\alpha_t^{(0)}$, G_0, reference temperature, heat-conduction coefficient, volume specific heat, temperature coefficient of linear expansion, and shear modulus, respectively.

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